Can wave–particle duality be based on the uncertainty relation?

Stephan Dürr and Gerhard Rempe[a)]
Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, 85748 Garching, Germany

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Wave and particle properties of a quantum object cannot be observed simultaneously. In particular, the fringe visibility in an interferometer is limited by the amount of which-way information which can be obtained. This limit is set by the recently discovered duality relation. So far, all derivations of the duality relation are independent of Heisenberg's uncertainty relation. Here we demonstrate that it is alternatively possible to derive the duality relation in the form of an uncertainty relation for some suitably chosen observables.

I. INTRODUCTION

Wave–particle duality refers to the fact that a quantum object can exhibit either wave or particle properties, depending on the experimental situation. In a double-slit experiment, for example, the object must pass through both slits simultaneously in order to create an interference pattern. This testifies to the object's wave nature. On the other hand, performing a which-way experiment reveals which of the slits each object passes through, manifesting its particle nature. However, performing a which-way experiment unavoidably destroys the interference pattern.

This was illustrated in various gedanken experiments, such as Einstein's recoiling slit[1] or Feynman's light microscope.[2] In order to explain the loss of interference in which-way experiments, one usually invokes Heisenberg's position–momentum uncertainty relation. This has been analyzed in great detail by, e.g., Wiseman *et al.*[3] However, Scully, Englert, and Walther[4] pointed out that such an explanation need not always be possible, but that the entanglement between the which-way marker and the interfering quantum object can always explain the loss of interference. Several experiments support this point of view.[5–11]

This entanglement need not always be perfect. In general, a measurement performed on the which-way marker yields only incomplete which-way knowledge. In order to quantify how much which-way information is available from such a measurement, one typically uses the ''distinguishability,'' D. With incomplete which-way information stored, one obtains interference fringes with a reduced visibility, V, which is limited by the so-called duality relation

$$D^2+V^2\leqslant 1. \tag{1}$$

This fundamental limit was recently discovered by Jaeger, Shimony, and Vaidman,[12] and independently by Englert.[13] It can be regarded as a quantitative statement about wave–particle duality. In the special case, where full which-way information is stored, $D=1$, it implies that the interference fringes are lost completely, $V=0$. The first experimental tests of the duality relation have been performed recently.[14,15]

Incomplete which-way information can alternatively be obtained without a which-way marker by setting up the interferometer such that the particle fluxes along the two ways differ. In this case, the which-way knowledge is expressed in the form of the so-called ''predictability,'' P, which is limited by[12,13,16–20]

$$P^2+V^2\leqslant 1. \tag{2}$$

This result was confirmed experimentally in Refs. 21 and 22.

None of the derivations of Eqs. (1) and (2) cited above involves any form of the uncertainty relation. It therefore seems that ''the duality relation is logically independent of the uncertainty relation.''[13] In this article, we will show, however, that for arbitrary which-way schemes, Eqs. (1) and (2) can always be derived in the form of a Heisenberg–Robertson uncertainty relation for some suitably chosen observables (which will turn out to be different from position and momentum).

II. PREDICTABILITY

In this section, we consider a two-beam interferometer without a which-way marker, as shown in Fig. 1. Let $|+\rangle$ and $|-\rangle$ denote the state vectors corresponding to the two ways along which the object can pass through the interferometer. After passing the first beam splitter, the density matrix in a representation with respect to the basis $\{|+\rangle,|-\rangle\}$ reads

$$\rho=\begin{pmatrix} w_+ & \rho_\pm \\ \rho_\pm^* & w_- \end{pmatrix}. \tag{3}$$

The probabilities w_+ and w_- that the object moves along one way or the other, respectively, fulfill $\mathrm{Tr}\{\rho\}=w_+ + w_- =1$. The magnitude of the difference between these probabilities is the predictability

$$P=|w_+ - w_-|, \tag{4}$$

which is obviously determined by the reflectivity of the first beam splitter. P quantifies how much which-way knowledge we have. For $P=0$, corresponding to a 50:50 beam splitter, we have no which-way knowledge, whereas for $P=1$, we know precisely which way the object takes.

Without loss of generality, we assume that the second beam splitter is a 50:50 beam splitter. Taking into account the phase shift φ between the two interferometer arms, the upper output beam corresponds to the state vector $|u_\varphi\rangle =(|+\rangle+e^{i\varphi}|-\rangle)/\sqrt{2}$. The intensity in this beam is

$$I_u(\varphi)\propto\langle u_\varphi|\rho|u_\varphi\rangle=\tfrac{1}{2}(1+2|\rho_\pm|\cos(\varphi+\varphi_0)) \tag{5}$$

with $\rho_\pm=|\rho_\pm|e^{i\varphi_0}$. The visibility of this interference pattern is

$$V=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=2|\rho_\pm|, \tag{6}$$

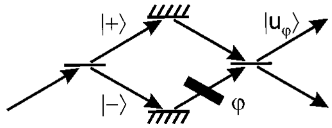

Fig. 1. Scheme of a typical two-beam interferometer. The incoming beam (left) is split into two beams, denoted $|+\rangle$ and $|-\rangle$. After reflection from mirrors, the phase of one of the beams is shifted by φ. Next, the two beams are recombined on a second beam splitter. Due to interference, the intensities of the two outgoing beams vary as a function of the phase shift φ.

where $I_{\max}$ and $I_{\min}$ denote the maximum and minimum intensities. The relation, Eq. (2), limiting visibility and predictability can easily be derived from $\mathrm{Tr}\{\rho^2\}=w_+^2+w_-^2+2|\rho_\pm|^2=\{1+P^2+V^2\}/2\leqslant 1$.

We will now show that this inequality can alternatively be obtained in the form of a Heisenberg–Robertson uncertainty relation[23,24]

$$\Delta A \Delta B \geqslant \tfrac{1}{2}|\langle [A,B]\rangle|, \tag{7}$$

which applies to each pair of Hermitian operators A and B, with the expectation values and standard deviations of operators defined as $\langle A\rangle=\mathrm{Tr}\{\rho A\}$ and $\Delta A=\sqrt{\langle A^2\rangle-\langle A\rangle^2}$, respectively.

In order to find suitable operators A and B, we investigate the Pauli spin-matrices

$$\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{8}$$

Their expectation values are $\langle\sigma_x\rangle=2\,\mathrm{Re}\{\rho_\pm\}$, $\langle\sigma_y\rangle=-2\,\mathrm{Im}\{\rho_\pm\}$, and $\langle\sigma_z\rangle=w_+-w_-$. Obviously, $\langle\sigma_z\rangle$ reflects our which-way knowledge, whereas $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ are related to the interference pattern via

$$I_u(\varphi)\propto \tfrac{1}{2}(1+\cos\varphi\langle\sigma_x\rangle+\sin\varphi\langle\sigma_y\rangle). \tag{9}$$

Without loss of generality, we choose the relative phase between states $|+\rangle$ and $|-\rangle$ such that $\rho_\pm$ is real, i.e., $\varphi_0=0$. Thus we obtain

$$|\langle\sigma_x\rangle|=V,\quad \langle\sigma_y\rangle=0,\quad |\langle\sigma_z\rangle|=P. \tag{10}$$

With this choice of the phases, $\langle\sigma_x\rangle$ and $\langle\sigma_z\rangle$ represent the wave character and particle character of the ensemble, respectively. The standard deviations of these observables,

$$\Delta\sigma_x=\sqrt{1-V^2},\quad \Delta\sigma_y=1,\quad \Delta\sigma_z=\sqrt{1-P^2}, \tag{11}$$

are easily obtained, because $\sigma_x^2=\sigma_y^2=\sigma_z^2=\mathbf{1}$. Using the commutator $[\sigma_j,\sigma_k]=2i\Sigma_l\epsilon_{jkl}\sigma_l$, we can now evaluate the uncertainty relation, Eq. (7), for all possible pairs of the above standard deviations, yielding

$$\sqrt{1-V^2}=\Delta\sigma_x\Delta\sigma_y\geqslant|\langle\sigma_z\rangle|=P, \tag{12}$$

$$\sqrt{1-P^2}=\Delta\sigma_y\Delta\sigma_z\geqslant|\langle\sigma_x\rangle|=V, \tag{13}$$

$$\Delta\sigma_z\Delta\sigma_x\geqslant|\langle\sigma_y\rangle|=0. \tag{14}$$

Equation (14) yields a trivial result, because standard deviations are non-negative by definition. However, Eqs. (12) and (13) are equivalent to the desired relation, Eq. (2). Hence, for the case without a which-way marker, Eq. (2) can be derived in the form of an uncertainty relation for the components of an abstract pseudospin.

III. DUALITY RELATION

Let us now add a second quantum system (called which-way marker) to the interferometer. When an object is passing through the interferometer, a suitable interaction shall change the quantum state of the which-way marker depending on the way the object took. This creates an entanglement between the which-way marker and the way of the object. A later measurement on the which-way marker can then reveal which way the object took. In other words, which-way information is now stored in the which-way marker. For simplicity, we assume that the which-way marker does not suffer from decoherence[25] (at least as long as we do not couple the marker to a macroscopic ''needle'').

Let ρ_{tot} denote the density matrix of the total system (object plus which-way marker) after the interaction (but before the phase shifter and the second beam splitter). Again, we denote the pseudospin corresponding to the ways by σ_x, σ_y, and σ_z. And again, we choose the relative phase between states $|+\rangle$ and $|-\rangle$ such that $\langle +|\mathrm{Tr}_M\{\rho_{\mathrm{tot}}\}|-\rangle$ is real, where Tr_M denotes the trace over the which-way marker. Thus we reproduce the above results, in particular,

$$|\langle\sigma_x\rangle|=V,\quad \Delta\sigma_x=\sqrt{1-V^2}. \tag{15}$$

In order to read out the which-way information, we measure an observable W of the which-way marker with eigenvalues $\{w_1,w_2,\ldots\}$ and an orthonormal basis of eigenstates $\{|w_1\rangle,|w_2\rangle,\ldots\}$. Let $p(\pm,w_i)$ denote the joint probability that w_i is found and that the object moves along way $|\pm\rangle$. If w_i is found, the best guess one can make about the way is to opt for way $|+\rangle$ if $p(+,w_i)\geqslant p(-,w_i)$, and for way $|-\rangle$ otherwise. This yields the ''likelihood for guessing the way right,''[13]

$$L_W=\sum_i \max\{p(+,w_i),p(-,w_i)\}. \tag{16}$$

Since L_W can vary between 1/2 and 1, it is natural to scale this quantity by defining the ''which-way knowledge''[26]

$$K_W=2L_W-1=\sum_i |p(+,w_i)-p(-,w_i)| \tag{17}$$

so that $0\leqslant K_W\leqslant 1$. Obviously, K_W depends on the choice of the measured observable W. In order to quantify how much which-way information is actually stored, the arbitrariness of the read-out process can be eliminated by defining the ''distinguishability''[12,13,26]

$$D=\max_W\{K_W\}, \tag{18}$$

which is the maximum value of K_W that is obtained for the best choice of W. The distinguishability is limited by the duality relation, Eq. (1), which has been derived in Refs. 12 and 13 without using the uncertainty relation.

We will now show that the duality relation—just as Eq. (2)—can alternatively be derived in the form of a Heisenberg–Robertson uncertainty relation for some suitably chosen observables. For that purpose, let

$$\epsilon_i=\begin{cases}+1 & \text{if } p(+,w_i)\geqslant p(-,w_i)\\ -1 & \text{otherwise}\end{cases} \tag{19}$$

denote which way to bet on if the eigenstate $|w_i\rangle$ is found. Using $p(+,w_i)=\langle w_i+|\rho_{\mathrm{tot}}|w_i+\rangle$, we thus find

$$K_W=\sum_i \epsilon_i(\langle w_i+|\rho_{\rm tot}|w_i+\rangle-\langle w_i-|\rho_{\rm tot}|w_i-\rangle) \qquad (20)$$

$$=\sum_i \epsilon_i {\rm Tr}\{\rho_{\rm tot}(|w_i\rangle\langle w_i|\otimes\sigma_z)\}, \qquad (21)$$

where we used $\sigma_z=|+\rangle\langle+|-|-\rangle\langle-|$ and where Tr denotes the trace over the total system. Let us define the observable

$$W_\epsilon=\sum_i \epsilon_i|w_i\rangle\langle w_i|. \qquad (22)$$

In passing, we note that $W_\epsilon^2=\mathbf{1}$ and $[\sigma_x,W_\epsilon]=[\sigma_y,W_\epsilon]=[\sigma_z,W_\epsilon]=0$. Inserting W_ϵ into Eq. (21), we obtain

$$K_W=\langle\sigma_z W_\epsilon\rangle. \qquad (23)$$

Note that we are considering a joint observable of the total system (object plus which-way marker) here, which is clearly necessary to explore the correlations between the which-way marker and the way taken by the object.

Let us now choose an observable $W_{\rm max}$, such that K_W is maximized. For simplicity, we will denote the corresponding observable defined by Eq. (22) by W_0 (instead of $W_{{\rm max},\epsilon}$). Hence, we obtain

$$D=\langle\sigma_z W_0\rangle. \qquad (24)$$

It is easy to see that $\sigma_z W_0$ is Hermitian and that $(\sigma_z W_0)^2=\mathbf{1}$, so that its standard deviation is

$$\Delta(\sigma_z W_0)=\sqrt{1-D^2}. \qquad (25)$$

Additionally, let us consider the observable $\sigma_y W_0$ which also fulfills $(\sigma_y W_0)^2=\mathbf{1}$. As it is also Hermitian, its expectation value is real, so that

$$\Delta(\sigma_y W_0)=\sqrt{1-\langle\sigma_y W_0\rangle^2}\leqslant 1. \qquad (26)$$

Using the commutator $[(\sigma_y W_0),(\sigma_z W_0)]=2i\sigma_x$, we can now write down the corresponding uncertainty relation. In combination with Eqs. (15), (25), and (26), we obtain

$$\sqrt{1-D^2}\geqslant\Delta(\sigma_y W_0)\Delta(\sigma_z W_0)\geqslant|\langle\sigma_x\rangle|=V. \qquad (27)$$

This directly yields the duality relation, Eq. (1). Alternatively, the commutator $[\sigma_x,(\sigma_y W_0)]=2i\sigma_z W_0$ can be used to obtain the uncertainty relation

$$\sqrt{1-V^2}\geqslant\Delta\sigma_x\Delta(\sigma_y W_0)\geqslant|\langle\sigma_z W_0\rangle|=D, \qquad (28)$$

which again yields the duality relation.

To summarize, we have demonstrated here that in an arbitrary which-way scheme, the duality relation can be expressed in the form of a Heisenberg–Robertson uncertainty relation for some suitably chosen observables.

IV. DISCUSSION

The above calculation reveals a new aspect of the connection between wave–particle duality and the uncertainty relation. We would like to add a few comments concerning the interpretation of this result.

Let us first point out that the uncertainty relation used in our calculation is not the position–momentum uncertainty relation. This is obvious, because, for example, the observables considered here have only two eigenvalues, namely ± 1, whereas position and momentum have a continuous spectrum of eigenvalues.

Second, we note that for the case without a which-way marker, Eq. (2) is *equivalent* to the uncertainty relations for $\Delta\sigma_x\Delta\sigma_y$ and $\Delta\sigma_y\Delta\sigma_z$, Eqs. (12) and (13). This equivalence can be read in both directions: In one direction, as discussed above, the uncertainty relation implies Eq. (2). In the other direction, Eq. (2) implies the uncertainty relation for these specific observables.

Third, we would like to draw attention to the fact that the uncertainty relation for $\Delta\sigma_z\Delta\sigma_x$, Eq. (14), yields a trivial result. This is somewhat surprising, because from Eq. (10) we concluded that σ_x represents the wave character, whereas σ_z represents the particle character. Since we are investigating the limit for the simultaneous presence of wave character and particle character, one might have guessed that the uncertainty relation for $\Delta\sigma_z\Delta\sigma_x$ could yield this limit. However, this is not the case. Instead, $\Delta\sigma_y$ is employed in our calculation. An intuitive interpretation of σ_y in terms of a wave picture or a particle picture is not obvious.

Next, we would like to mention that the observables whose uncertainty relations we evaluate in Eqs. (12) and (13) depend on the density matrix, ρ. In the presentation in Sec. II, this fact is somewhat hidden in our choice of the relative phase of states $|+\rangle$ and $|-\rangle$, i.e., $\varphi_0=0$. The dependence on ρ becomes more obvious, if we consider arbitrary values of φ_0. In this case, we can define the observables

$$\Sigma_x=\sigma_x\cos\varphi_0-\sigma_y\sin\varphi_0, \qquad (29)$$

$$\Sigma_y=\sigma_x\sin\varphi_0+\sigma_y\cos\varphi_0, \qquad (30)$$

$$\Sigma_z=\sigma_z, \qquad (31)$$

which take the role of σ_x, σ_y, and σ_z in our above presentation. Obviously, these observables depend on ρ via φ_0. As the commutation relations of the Σ's and σ's are the same, Eq. (2) can be derived analogously. The situation is similar in Sec. III.

Finally, we will discuss whether either correlations (i.e., entanglement) or uncertainty relations are more closely connected to wave–particle duality. For that purpose, we will investigate all the explanations for the loss of interference fringes, referenced in Sec. I. We will sort these explanations into three categories, depending on whether they employ

(1) some uncertainty relation,
(2) correlations,
(3) correlations and some uncertainty relation.

The textbook explanations for Einstein's recoiling slit in Ref. 1 and Feynman's light microscope in Ref. 2 are based on the position–momentum uncertainty relation. The Scully–Englert–Walther explanation[4] as well as the derivations of the duality relation in Refs. 12 and 13 are based on the correlations. Our derivation as well as the discussion of Wiseman *et al.*[3] make use of both the correlations and some uncertainty relation. This is because these calculations involve the density matrix for the total system, consisting of the object plus the which-way marker. Consequently, the full quantum correlations between these subsystems are embodied in these formalisms.

The above categorization reveals a crucial point: The explanations for the loss of interference fringes involving *only* the uncertainty relation are (so far) limited to a few special schemes. In other words: There are several other schemes for which no such explanation is known, see, e.g., Refs. 4 and

11. In the language of Ref. 3, the loss of interference in these schemes cannot be explained in terms of ''classical momentum transfer.'' On the other hand, explanations involving *only* correlations apply to all which-way schemes known so far. This leads us to the conclusion that wave–particle duality is connected to correlations more closely than to the uncertainty relation.

[a)]Electronic mail: gerhard.rempe@mpq.mpg.de
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